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On the weak amenability of $\mathscr{A}(X)$ and its relation with the approximation property ☆

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Abstract

We investigate the weak amenability of the algebra $\mathscr{A}(X)$ of approximable operators on a Banach space X, and its relation with the (bounded) approximation property. In particular, it will be shown that the (bounded) approximation property is neither necessary nor sufficient for the weak amenability of $\mathscr{A}(X)$.

MSC: 46B28; 47L10

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1. Introduction

Let $\mathfrak{A}$ be a Banach algebra, and let $\mathfrak{X}$ be a Banach $\mathfrak{A}$-bimodule. A (bounded) linear map $D : \mathfrak{A} \to \mathfrak{X}$ that satisfies the identity

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathfrak{A}).$$

is called a (continuous) derivation. Every map of the form $a \mapsto a \cdot x - x \cdot a$ ($a \in \mathfrak{A}$), where $x \in \mathfrak{X}$ is fixed, is obviously a continuous derivation. Derivations of this form are called inner derivations.

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The first Hochschild–Johnson cohomology group of $\mathfrak{A}$ with coefficients in an $\mathfrak{A}$-bimodule $\mathfrak{X}$, denoted by $\mathscr{H}^1(\mathfrak{A},\mathfrak{X})$, is defined as the quotient of the space of continuous derivations from $\mathfrak{A}$ into $\mathfrak{X}$ by the corresponding (sub)space of inner derivations. Thus, triviality of $\mathscr{H}^1(\mathfrak{A},\mathfrak{X})$ amounts to every continuous derivation from $\mathfrak{A}$ into $\mathfrak{X}$ being inner.

The topological dual $\mathfrak{X}'$ of a Banach $\mathfrak{A}$-bimodule $\mathfrak{X}$ is also a Banach $\mathfrak{A}$-bimodule under the actions

$$(a\cdot f)(x) = f(xa) \quad \text{and} \quad (f\cdot a)(x) = f(ax) \qquad (a\in\mathfrak{A}, x\in\mathfrak{X}, f\in\mathfrak{X}').$$

A Banach algebra $\mathfrak{A}$ is said to be *amenable* if, for every Banach $\mathfrak{A}$-bimodule $\mathfrak{X}$, $\mathscr{H}^1(\mathfrak{A},\mathfrak{X}') = \{0\}$ [J1], and *weakly amenable* if $\mathscr{H}^1(\mathfrak{A},\mathfrak{A}') = \{0\}$ [BCD,J2].

For instance, the group algebra, $L^1(G)$, of a locally compact group G is always weakly amenable [J3], and is amenable if and only if G is amenable in the classical sense [J1]; a C^*-algebra is always weakly amenable [Ha], and is amenable if and only if it is nuclear [Co,Ha].

In this paper we shall be concerned with the weak amenability of the algebra $\mathscr{A}(X)$ of *approximable operators* on a Banach space X. Here, $\mathscr{A}(X)$ is the closure in $\mathscr{B}(X)$ of the ideal $\mathscr{F}(X)$ of *continuous finite-rank operators* on X, where $\mathscr{B}(X)$ denotes, as usual, the algebra of all *bounded linear operators* on X. Special emphasis will be put on the relationship between weak amenability of this algebra and the (bounded) approximation property.

Recall that a Banach space X is said to have the *approximation property* (AP in short), if for every compact set $K\subset X$ and every $\epsilon>0$, there is an operator $T\in\mathscr{F}(X)$ such that $\|Tx-x\|\leqslant\epsilon$ for every x in K. If, in addition, the operator T can always be chosen with $\|T\|\leqslant\lambda$ for some $\lambda\geqslant 1$ fixed, then X is said to have the *λ-approximation property* (λ-AP in short). A Banach space X is said to have the *bounded approximation property* (BAP in short) if it has the λ-AP for some λ.

The study of the amenability of $\mathscr{A}(X)$ goes back to the origins of the subject in [J1]. Further progress was made in [GJW], where amenability of $\mathscr{A}(X)$ is described as a (symmetric) approximation property. Actually X' (and hence X) has BAP whenever $\mathscr{A}(X)$ is amenable.

Some results on weak amenability of $\mathscr{A}(X)$ are obtained in [DGG]. It is shown there that $\mathscr{A}(X)$ is weakly amenable whenever X has one of the following forms: (i) $X = E\oplus C_p$, where C_p denotes any of the universal spaces introduced by Johnson [Joh] and E is any Banach space with BAP; and (ii) $X = l_p(Y)$, $1<p<\infty$, where Y is any reflexive Banach space with AP.

All the above results were good reasons to suspect that BAP would be important in trying to explain the weak amenability of $\mathscr{A}(X)$ as a consequence of the geometry of X. The main outcome of this paper is that the weak amenability of $\mathscr{A}(X)$ is largely independent of whether or not X has the BAP.

The paper has been organized as follows. First, in Section 2 we shall give a characterization of the weak amenability of $\mathscr{A}(X)$. This characterization will be essential for the results of the subsequent sections.

In Section 3 we shall investigate the weak amenability of $\mathscr{A}(X)$ in the context of direct sums, duals, and predual spaces. The BAP will play a significant role in most of the results of this section. In particular, it will be shown that, contrary to what happens with amenability, the class of Banach spaces X with BAP for which $\mathscr{A}(X)$ is weakly amenable is invariant under direct sums. Also as a simple consequence of our results on direct sums it will be seen that (AP) BAP is not required for weak amenability.

The failure of BAP to be a necessary condition is the main motivation for the results of Section 4. In this section a necessary condition will be given and, using it, examples will be constructed of Banach spaces X for which $\mathscr{A}(X)$ is not weakly amenable. All the examples given in this section will have the additional feature of failing AP. Thus, these results will not allow us to draw any conclusion about the sufficiency of BAP for the weak amenability of $\mathscr{A}(X)$.

The question of whether or not BAP is sufficient will be answered, also in the negative, in the last section of this paper. Concrete examples will be provided of Banach spaces X with BAP such that $\mathscr{A}(X)$ is not weakly amenable. Sufficient conditions for the weak amenability of $\mathscr{A}(X)$ can be found in [Bl].

We shall assume throughout that all our Banach spaces are over the complex field.

2. A characterization of weak amenability of the algebra $\mathscr{A}(E)$

We start by establishing some terminology.

Given a Banach space E, we identify $\mathscr{F}(E)$ and $E' \otimes E$ in the usual way, that is, we associate to the element $v = \sum_i \lambda_i \otimes x_i \in E' \otimes E$ the operator $\bar{v} \in \mathscr{F}(E)$ defined by $\bar{v}(x) = \sum_i \lambda_i(x) x_i$ $(x \in E)$. In particular, we shall talk about the operator norm (instead of the injective norm) of an element in $E' \otimes E$, and the projective norm of an operator in $\mathscr{F}(E)$. We denote the latter by $\|\cdot\|_\wedge$.

The completion of $E' \otimes E$ $(= \mathscr{F}(E))$ in the projective norm is the *tensor algebra* of E and is denoted by $E' \hat{\otimes} E$. It is well known (see [He, II. 2.19]) that $E' \hat{\otimes} E$ is a Banach algebra.

The canonical trace on $E' \hat{\otimes} E$, denoted by tr_E (or just tr if the space E is clear from the context), is the unique, continuous linear functional on $E' \hat{\otimes} E$, which is defined on elementary tensors by

$$\mathrm{tr}_E(\lambda \otimes x) = \lambda(x) \quad (\lambda \in E', x \in E).$$

Given a finite-rank operator W, we denote by $\mathrm{tr}\, W$, the usual (operator) trace. The latter will not cause any trouble with our previous conventions, as the canonical trace of an element $v \in E' \otimes E$ coincides with the trace, in the usual sense, of the associated operator $\bar{v}$ in $\mathscr{F}(E)$.

The adjoint of a bounded operator U is denoted by U'.

If X and Y are isomorphic (respectively, isometric) normed spaces, we write this as $X \simeq Y$ (respectively, $X \cong Y$), and denote by $d(X, Y)$ the Banach–Mazur distance between them, that is, the infimum of numbers $||T||\,||T^{-1}||$, where T is an isomorphism between X and Y.

The completion of a normed space X is denoted by X^{-}. If $x_1, x_2, \dots, x_r$ are vectors of some linear space X, we denote by $\mathrm{sp}\{x_1, x_2, \dots, x_r\}$ their linear span.

Given a normed linear algebra $\mathfrak{A}$, and an $\mathfrak{A}$-bimodule $\mathfrak{X}$, we denote by $\mathscr{Z}(\mathfrak{X})$ its centre.

The main result of this section is the following.

Theorem 2.1. *Let E be a Banach space, and let $\mathfrak{A}$ be a dense subalgebra of $(E' \otimes E, ||\cdot||_{\wedge})$ (and hence of $E' \hat{\otimes} E$). The algebra $\mathscr{A}(E)$ is weakly amenable if and only if, whenever $T \in \mathscr{B}(E')$ satisfies*

$$|\mathrm{tr}(T(RS - SR)')| \leqslant K||R||\,||S|| \quad (R, S \in \mathfrak{A})$$

for some constant K, the following holds:

(A) *There exists $\lambda \in \mathbb{C}$ and a constant $\tilde{K}$ such that*

$$|\mathrm{tr}((T - \lambda)W')| \leqslant \tilde{K}||W|| \quad (W \in \mathfrak{A}).$$

Moreover, (A) *is equivalent to the following condition:*

(A*) *there exists a constant K_o such that*

$$|\mathrm{tr}(TW')| \leqslant K_o||W||$$

for all $W \in \mathfrak{A}$ such that $\mathrm{tr}\, W = 0$.

In proving Theorem 2.1, we shall need the next two lemmas.

Lemma 2.2. *Let $\mathfrak{A}_1 = (\mathfrak{A}, ||\cdot||_1)$ and $\mathfrak{A}_2 = (\mathfrak{A}, ||\,.\,||_2)$ be normed algebras such that $||\,.\,||_1 \geqslant ||\,.\,||_2$, let $(\mathfrak{X}_1, |||\,.\,|||_1)$ be a Banach $\mathfrak{A}_1$-bimodule, and let $(\mathfrak{X}_2, |||\,.\,|||_2)$ be a Banach $\mathfrak{A}_2$-bimodule. Moreover, suppose that $\mathfrak{X}_2$ is contained in $\mathfrak{X}_1$ with $|||\,.\,|||_2 \geqslant |||\,.\,|||_1$. If $D : \mathfrak{A}_2 \to \mathfrak{X}_2$ is a continuous derivation, then $\partial : \mathfrak{A}_1 \to \mathfrak{X}_1$ defined by $\partial := \imath \circ D \circ I_{\mathfrak{A}}$, where $\imath : \mathfrak{X}_2 \to \mathfrak{X}_1$ denotes the natural inclusion, and $I_{\mathfrak{A}} : \mathfrak{A}_1 \to \mathfrak{A}_2$ denotes the formal identity map, is also a continuous derivation. In particular, we may take $\mathfrak{X}_i = \mathfrak{A}_i'$ $(i = 1, 2)$.*

Proof. This is immediate from the definitions. $\square$

Lemma 2.3. *Let E be an infinite-dimensional Banach space, let $\mathfrak{A}$ be a dense subalgebra of $(E' \otimes E, ||\,.\,||_{\wedge})$, and let $T \in \mathscr{B}(E')$. Then there exist $\lambda \in \mathbb{C}$ and a bounded sequence $(\Lambda_n) \subset \mathfrak{A}$ with* $\mathrm{tr}(\Lambda_n) = n$ $(n \in \mathbb{N})$ *such that*

$$\sup_n\{|\mathrm{tr}(T\Lambda_n') - \lambda n|\} < \infty. \tag{1}$$

Proof. By the well-known theorem of Dvoretsky on spherical sections of convex bodies, for each $n \in \mathbb{N}$ there is an n-dimensional subspace $E_n \subset E$ such that $d(E_n, l_2^n) < 2$. Let $T_n : E_n \to l_2^n$ be a linear isomorphism such that $\|T_n\|\,\|T_n^{-1}\| \leqslant 2$, let $\{e_1, e_2, \dots, e_n\}$ be the unit vector basis of l_2^n, and let $\{e_1^*, e_2^*, \dots, e_n^*\}$ be the corresponding set of biorthogonal functionals ($n \in \mathbb{N}$). Define

$$x_{i,n} \coloneqq T_n^{-1} e_i \quad \text{and} \quad x_{i,n}^* \coloneqq T_n' e_i^* \quad (1 \leqslant i \leqslant n, n \in \mathbb{N}),$$

and let $\lambda_{i,n} \in E'$ be an extension of $x_{i,n}^*$ such that $\|\lambda_{i,n}\| = \|x_{i,n}^*\|$ $(1 \leqslant i \leqslant n, n \in \mathbb{N})$. Then define the projection

$$S_n \coloneqq \sum_{i=1}^{n} \lambda_{i,n} \otimes x_{i,n} \quad (n \in \mathbb{N}).$$

It is easily verified that the sequence (S_n) satisfies: (i) $\operatorname{tr} S_n = n$, (ii) $\|S_n\| \leqslant 2\sqrt{n} = o(\operatorname{tr} S_n)$, and (iii) $\|S_n\|_\wedge \leqslant 2n = O(\operatorname{tr} S_n)$ $(n \in \mathbb{N})$.

Since $\mathfrak{A}$ is dense in $\mathscr{F}(E)$ in the projective norm, there exists $(\tilde{S}_n) \subset \mathfrak{A}$ such that $\|S_n - \tilde{S}_n\|_\wedge \leqslant 1$ $(n \in \mathbb{N})$. Then $|\operatorname{tr} S_n - \operatorname{tr} \tilde{S}_n| \leqslant 1$ $(n \in \mathbb{N})$, and consequently $\|\tilde{S}_n\| \leqslant o(\operatorname{tr} S_n) + 1 = o(\operatorname{tr} \tilde{S}_n)$ and $\|\tilde{S}_n\|_\wedge = O(\operatorname{tr} S_n) + 1 = O(\operatorname{tr} \tilde{S}_n)$.

Let $(\lambda_n) \subset \mathbb{C}$ be the sequence defined by $\lambda_n \coloneqq \operatorname{tr}(T\tilde{S}_n')/\operatorname{tr} \tilde{S}_n$ $(n \in \mathbb{N})$. It is easy to see that (λ_n) is bounded. In fact, we have

$$|\operatorname{tr}(T\tilde{S}_n')| \leqslant \|T\|\,\|\tilde{S}_n'\|_\wedge \leqslant \|T\|\,\|\tilde{S}_n\|_\wedge = O(\operatorname{tr} \tilde{S}_n) \quad (n \in \mathbb{N}).$$

Let $(\tilde{S}_{n_k})$ be a subsequence of $(\tilde{S}_n)$ such that $\lim_k k(\lambda_{n_k} - \lambda) = 0$ (for some $\lambda \in \mathbb{C}$) and $k\|\tilde{S}_{n_k}\| = O(\operatorname{tr} \tilde{S}_{n_k})$ (such a sequence exists, since (λ_n) is bounded and $\|\tilde{S}_n\| = o(\operatorname{tr} \tilde{S}_n)$). Then the sequence $(\Lambda_k) \subset \mathfrak{A}$ defined by $\Lambda_k \coloneqq k\tilde{S}_{n_k}/\operatorname{tr} \tilde{S}_{n_k}$ $(k \in \mathbb{N})$ has all the required properties. □

Proof of Theorem 2.1. We start by proving that $\mathscr{A}(E)$ is weakly amenable if and only if (A) is satisfied.

Let $\mathfrak{A}$ be as in the hypotheses. Define $\mathfrak{A}_1 \coloneqq (\mathfrak{A}, \|\,.\,\|_\wedge)$ and $\mathfrak{A}_2 \coloneqq (\mathfrak{A}, \|\,.\,\|)$. It is easily seen that $\mathfrak{A}_1^- \cong E' \hat{\otimes} E$ and $\mathfrak{A}_2^- \cong \mathscr{A}(E)$. Then note that condition (A) is equivalent to the following:

(†) if $f \in \mathfrak{A}_1'$ is such that the map $S \mapsto S \cdot f - f \cdot S$ $(S \in \mathfrak{A})$ defines a continuous derivation from $\mathfrak{A}_2$ into $\mathfrak{A}_2'$, then there is $g \in \mathfrak{A}_2'$ such that $f - g \in \mathscr{Z}(\mathfrak{A}_1')$.

To see this, for every $T \in \mathscr{B}(E')$, define $T^\wedge : \mathfrak{A}_1 \to \mathbb{C}$ by $T^\wedge(S) \coloneqq \operatorname{tr}(TS')$ $(S \in \mathfrak{A})$. The map $T \mapsto T^\wedge$, $\mathscr{B}(E') \to \mathfrak{A}_1'$, is an isometric linear isomorphism [Pa, 1.7.11]. Thus, given $f \in \mathfrak{A}_1'$, there exists a unique $T \in \mathscr{B}(E')$ such that $T^\wedge = f$, and so for $R, S \in \mathfrak{A}$,

$$\begin{aligned}(S \cdot f - f \cdot S)(R) &= (S \cdot T^\wedge - T^\wedge \cdot S)(R) \\ &= T^\wedge(RS - SR) = \operatorname{tr}(T(RS - SR)'). \qquad (2)\end{aligned}$$

The continuity of $S \mapsto S \cdot f - f \cdot S$ $(S \in \mathfrak{A})$ as a map from $\mathfrak{A}_2$ into $\mathfrak{A}_2'$ means that

$$||S \cdot f - f \cdot S|| \leqslant K_f ||S|| \quad (S \in \mathfrak{A}_2) \tag{3}$$

for some constant K_f. Combining (2) and (3), we find that, if $S \mapsto S \cdot f - f \cdot S$ $(S \in \mathfrak{A})$ defines a continuous derivation from $\mathfrak{A}_2$ into $\mathfrak{A}_2'$, then

$$\begin{aligned} |\mathrm{tr}(T(RS - SR)')| &= |(S \cdot f - f \cdot S)(R)| \\ &\leqslant K_f ||S|| \, ||R|| \quad (R, S \in \mathfrak{A}). \end{aligned}$$

On the other hand, by [Gr1, Lemma 2], $\mathscr{Z}(\mathfrak{A}_1') = \mathrm{sp}\{\mathrm{tr}\}$ and so we have $f - g \in \mathscr{Z}(\mathfrak{A}_1')$ with $g \in \mathfrak{A}_2'$ if and only if $f - \lambda \mathrm{tr} \in \mathfrak{A}_2'$ for some $\lambda \in \mathbb{C}$. Moreover,

$$\begin{aligned} (f - \lambda \mathrm{tr})(W) &= (T^\wedge - \lambda I_E^\wedge)(W) \\ &= \mathrm{tr}((T - \lambda I_E) W') \quad (W \in \mathfrak{A}). \end{aligned}$$

This shows that (A) and (†) are equivalent.

Next, we show that $\mathscr{A}(E)$ is weakly amenable if and only if (†) is satisfied.

Assume that $\mathfrak{A}_2^-$ is weakly amenable. Let $f \in \mathfrak{A}_1'$ be such that the map $D : \mathfrak{A}_2 \to \mathfrak{A}_2'$, $S \mapsto S \cdot f - f \cdot S$ is a continuous derivation. Then D must be inner, that is, $D(S) = S \cdot g - g \cdot S$ $(S \in \mathfrak{A})$ for some $g \in \mathfrak{A}_2'$. It follows from this and the definition of D that $f - g \in \mathscr{Z}(\mathfrak{A}_1')$. Thus, (†) (and hence (A)) is satisfied. This proves the 'only if' part.

Conversely, assume (†) holds. Let $D : \mathfrak{A}_2 \to \mathfrak{A}_2'$ be a continuous derivation. By Lemma 2.2, with $X_i = \mathfrak{A}_i'$ $(i = 1, 2)$, the operator $\partial := \imath \circ D \circ I_{\mathfrak{A}} : \mathfrak{A}_1 \to \mathfrak{A}_1'$ is also a continuous derivation. Since $\mathfrak{A}_1^- \cong E' \hat{\otimes} E$ and the tensor algebra of E is weakly amenable [DGG, Theorem 5.1], the continuous extension of ∂ to $\mathfrak{A}_1^-$ is an inner derivation. Thus $D(S) = \partial(S) = S \cdot f - f \cdot S$ $(S \in \mathfrak{A})$ for some $f \in \mathfrak{A}_1'$. Now (†) guarantees the existence of $g \in \mathfrak{A}_2'$ such that $f - g \in \mathscr{Z}(\mathfrak{A}_1')$. Equivalently $S \cdot f - f \cdot S = S \cdot g - g \cdot S$ $(S \in \mathfrak{A})$. The last condition clearly means that D is an inner derivation. Thus $\mathfrak{A}_2^-$ (and hence $\mathscr{A}(E)$) is weakly amenable. This proves the 'if' part.

We have shown that $\mathscr{A}(E)$ is weakly amenable if and only if (A) is satisfied. Let us now show that condition (A) is equivalent to (A*).

That (A) implies (A*) is obvious.

Let us prove the converse implication. Let T satisfy (A*). By Lemma 2.3, there exist $\lambda \in \mathbb{C}$ and a bounded sequence $(\Lambda_n) \subset \mathfrak{A}$ with $\mathrm{tr}(\Lambda_n) = n$ $(n \in \mathbb{N})$ such that (1) is satisfied. We show that, if (A*) holds, then condition (A) is satisfied for this value of λ. In fact, assume towards a contradiction that (A*) holds yet there exists a bounded sequence $(W_n) \subset \mathfrak{A}$ such that

$$\sup_n \{|\mathrm{tr}((T - \lambda) W_n')|\} = \infty.$$

Let $\mathrm{tr}(W_n) = \rho_n$. Without loss of generality we may suppose that ρ_n is a positive integer ($n \in \mathbb{N}$). Then

$$|\mathrm{tr}((T-\lambda)W_n')| \leqslant |\mathrm{tr}(T(W_n - \Lambda_{\rho_n})') - \lambda\mathrm{tr}((W_n - \Lambda_{\rho_n})')| \\ + |\mathrm{tr}(T\Lambda'_{\rho_n}) - \lambda\mathrm{tr}(\Lambda'_{\rho_n})|.$$

Both summands on the right-hand side of the above inequality are bounded, the first, because of (A*), and the second, because of our choice of λ and (Λ_n). The contradiction is now obvious. $\square$

Corollary 2.4. *Let E be a reflexive Banach space, and let $\mathfrak{A}$ be as in Theorem* 2.1. *Then $\mathscr{A}(E)$ is weakly amenable if and only if, whenever $T \in \mathscr{B}(E)$ is such that*

$$|\mathrm{tr}(T(RS - SR))| \leqslant K||R||\,||S|| \quad (R, S \in \mathfrak{A})$$

for some constant K, the following holds:

(A) *there exists $\lambda \in \mathbb{C}$ and a constant $\tilde{K}$ such that*

$$|\mathrm{tr}((T-\lambda)W)| \leqslant \tilde{K}||W|| \quad (W \in \mathfrak{A}).$$

Proof. When E is reflexive the map $T \mapsto T'$, $\mathscr{B}(E) \to \mathscr{B}(E')$, is an isometric anti-isomorphism. Using this fact, elementary properties of the trace, and Theorem 2.1, the desired result follows. $\square$

3. Direct sums, duals, and preduals

Given a direct sum of Banach spaces, $X = X_1 \oplus X_2$, we denote by $\gamma_k : X \to X_k$ ($\iota_k : X_k \to X$), $k = 1, 2$, the canonical kth coordinate projection (embedding). Then for $V \in \mathscr{B}(X)$ (respectively, $\tilde{V} \in \mathscr{B}(X')$) we denote by V_{kj} (respectively, $\tilde{V}_{kj}$) the operator $\gamma_k V \iota_j$ (respectively, $\iota_k' \tilde{V} \gamma_j'$), $1 \leqslant k, j \leqslant 2$. Without loss of generality, we shall suppose that the norm on $X_1 \oplus X_2$ has been chosen such that $||\gamma_k|| = 1$ for $k = 1, 2$.

For each $T \in \mathscr{B}(X')$, set

$$l_T : W \mapsto \mathrm{tr}(TW') \quad (W \in \mathscr{F}(X)) \tag{4}$$

and

$$b_T : (R, S) \mapsto \mathrm{tr}(T(RS - SR)') \quad ((R, S) \in \mathscr{F}(X) \times \mathscr{F}(X)). \tag{5}$$

Then define

$$\Delta_X := \{T \in \mathscr{B}(X') : b_T \text{ is bounded}$$
$$(\mathscr{F}(X) \text{ with the operator norm})\}. \tag{6}$$

We denote by $||b_T||$ the norm of b_T $(T \in \Delta_X)$.

Lemma 3.1. *Let X_1 and X_2 be infinite-dimensional Banach spaces, and let $X = X_1 \oplus X_2$. Suppose that at least one of the following conditions is satisfied*:

(i) *X_1' has the BAP*;
(ii) *X_2' has the BAP*;
(iii) *X has the BAP.*

Then, for each $T \in \Delta_X$, there exists a constant K_T such that

$$|l_T(\imath_1 W_{12}\gamma_2 + \imath_2 W_{21}\gamma_1)| \leqslant K_T ||W|| \quad (W \in \mathscr{F}(X)). \tag{7}$$

Proof. Let us suppose that X_1' has the λ-AP for some $\lambda \geqslant 1$. Then by [DF, Section 16.3, Corollary 1] X_1 has the λ-AP as well. Let $T \in \Delta_X$, and let $W \in \mathscr{F}(X)$. Take $P \in \mathscr{F}(X_1)$ with $||P|| \leqslant \lambda + 1$ such that $PW_{12} = W_{12}$, and let $Q \in \mathscr{F}(X_1)$ with $||Q|| \leqslant \lambda + 1$ such that $W_{21}Q = W_{21}$ (the existence of P and Q is guaranteed by [DF, Section 16.9, Corollary]). Then since $\gamma_1\imath_1 = I_{X_1}$, $\gamma_2\imath_1 = 0$ and $\gamma_1\imath_2 = 0$, we have

$$\begin{aligned}|l_T(\imath_1 W_{12}\gamma_2)| &= |l_T(\imath_1 P\gamma_1\imath_1 W_{12}\gamma_2 - \imath_1 W_{12}\gamma_2\imath_1 P\gamma_1)| \\ &= |b_T(\imath_1 P\gamma_1, \imath_1 W_{12}\gamma_2)| \leqslant ||b_T||\, ||\imath_1 P\gamma_1||\, ||\imath_1 W_{12}\gamma_2|| \\ &\leqslant (\lambda+1)||b_T||\, ||W||\end{aligned}$$

and

$$\begin{aligned}|l_T(\imath_2 W_{21}\gamma_1)| &= |l_T(\imath_2 W_{21}\gamma_1\imath_1 Q\gamma_1 - \imath_1 Q\gamma_1\imath_2 W_{21}\gamma_1)| \\ &= |b_T(\imath_1 Q\gamma_1, \imath_2 W_{21}\gamma_1)| \leqslant ||b_T||\, ||\imath_1 Q\gamma_1||\, ||\imath_2 W_{21}\gamma_1|| \\ &\leqslant (\lambda+1)||b_T||\, ||W||.\end{aligned}$$

Combining the last two inequalities, we obtain the desired one with $K_T = 2(\lambda+1)||b_T||$.
The proof is completely analogous if one of condition (ii) or (iii) be satisfied instead of (i). $\square$

Given a Banach space X and a continuous finite-rank operator W on a Banach space Y, we set

$$|W|_X = \inf\{||R||\, ||S|| : R \in \mathscr{F}(X, Y), S \in \mathscr{F}(Y, X) \text{ and } RS = W\}, \tag{8}$$

where $\mathscr{F}(X,Y)$ (respectively, $\mathscr{F}(Y,X)$) denotes the normed space of continuous finite-rank operators from X (respectively, Y) into Y (respectively, X).

Proposition 3.2. *Suppose that the hypotheses of the previous lemma are satisfied. If $\mathscr{A}(X_1)$ is weakly amenable and there exists $\rho > 0$ such that $|W|_{X_1} \leqslant \rho ||W||$ $(W \in \mathscr{F}(X_2))$, then $\mathscr{A}(X)$ is weakly amenable.*

Proof. We shall use the characterization given in Theorem 2.1.

Let $T \in \Delta_X$ and $W \in \mathscr{F}(X)$ be such that tr $W = 0$. It is easily seen that $T_{11} \in \Delta_{X_1}$. Since $\mathscr{A}(X_1)$ is weakly amenable, by Theorem 2.1, there exists $\lambda \in \mathbb{C}$ and a constant $K_{1T} > 0$ such that

$$|\mathrm{tr}((T_{11} - \lambda)\tilde{W}')| \leqslant K_{1T} ||\tilde{W}|| \quad (\tilde{W} \in \mathscr{F}(X_1)). \tag{9}$$

Taking into account the fact that tr W_{11} + tr W_{22} = tr $W = 0$, we see that

$$\begin{aligned} |l_T(\imath_1 W_{11}\gamma_1 + \imath_2 W_{22}\gamma_2)| &= |l_{T_{11}}(W_{11}) + l_{T_{22}}(W_{22})| \\ &\leqslant K_{1T}||W_{11}|| + |\mathrm{tr}((T_{22} - \lambda)W_{22}')|. \end{aligned} \tag{10}$$

We show next that there exists a constant $K_{2T} > 0$ independent of W such that

$$|\mathrm{tr}((T_{22} - \lambda)W_{22}')| \leqslant K_{2T}||W_{22}||.$$

Let $R \in \mathscr{F}(X_1, X_2)$ and $S \in \mathscr{F}(X_2, X_1)$. We have

$$\begin{aligned} &|\mathrm{tr}(T_{11}(SR)') - \mathrm{tr}(T_{22}(RS)')| \\ &\quad = |\mathrm{tr}(T(\imath_1 S\gamma_2\imath_2 R\gamma_1 - \imath_2 R\gamma_1\imath_1 S\gamma_2)')| \\ &\quad = |b_T(\imath_1 S\gamma_2, \imath_2 R\gamma_1)| \leqslant ||b_T||\, ||\imath_1 S\gamma_2||\, ||\imath_2 R\gamma_1|| \\ &\quad \leqslant ||b_T||\, ||R||\, ||S||. \end{aligned}$$

Using the last inequality and (9), we see that

$$\begin{aligned} &|\mathrm{tr}((T_{22} - \lambda)(RS)')| \\ &\quad \leqslant |\mathrm{tr}((T_{11} - \lambda)(SR)')| + |\mathrm{tr}(T_{11}(SR)') - \mathrm{tr}(T_{22}(RS)')| \\ &\quad \leqslant (K_{1T} + ||b_T||)||R||\, ||S|| \quad (R \in \mathscr{F}(X_1, X_2), S \in \mathscr{F}(X_2, X_1)). \end{aligned}$$

Then, taking into account the definition of $|\,.\,|_{X_1}$ (see (8)), it follows that

$$|\mathrm{tr}((T_{22} - \lambda)W_{22}')| \leqslant (K_{1T} + ||b_T||)|W_{22}|_{X_1},$$

and, in turn, by our hypothesis about $|\,.\,|_{X_1}$, that

$$|\mathrm{tr}((T_{22}-\lambda)W'_{22})| \leqslant K_{2T}\|W_{22}\|, \tag{11}$$

where $K_{2T} = \rho(K_{1T} + \|b_T\|)$.

Combining (7), (10), and (11), we see that

$$\begin{aligned} |l_T(W)| &\leqslant |l_T(\imath_1 W_{11}\gamma_1 + \imath_2 W_{22}\gamma_2)| + |l_T(\imath_1 W_{12}\gamma_2 + \imath_2 W_{21}\gamma_1)| \\ &\leqslant K_{1T}\|W_{11}\| + K_{2T}\|W_{22}\| + K_T\|W\| \leqslant \tilde{K}_T\|W\|, \end{aligned}$$

where $\tilde{K}_T = K_{1T} + K_{2T} + K_T$ (K_T as in Lemma 3.1). Thus (A*) is satisfied, and so, by Theorem 2.1, $\mathscr{A}(X)$ is weakly amenable. $\square$

Let (G_n) be a sequence of finite-dimensional Banach spaces dense (in the Banach–Mazur sense) in the class of all finite-dimensional Banach spaces, such that for every $n \in \mathbb{N}$ the set $\{i : G_n \cong G_i\}$ is infinite. The Banach space C_p ($p = 0$ or $1 \leqslant p \leqslant \infty$) is the l_p-sum of the sequence (G_n), that is, $C_p := (\sum \oplus_{n=1}^{\infty} G_n)_{l_p}$ (see [Joh,Joh1]).

Corollary 3.3. *For every Banach space E, the algebra $\mathscr{A}(E \oplus C_p)$ is weakly amenable.*

Proof. This is immediate from the above proposition and the definition of the Banach spaces C_p (recall that $\mathscr{A}(C_p)$, $p = 0$ or $1 \leqslant p < \infty$, is weakly amenable [Gr2]). $\square$

Corollary 3.4. *For every Banach space E, the algebra $\mathscr{A}(E \oplus C_\infty)$ is weakly amenable.*

Proof. The Banach space C_∞ contains a 1-complemented isometric copy of the dual of every separable Banach space [Joh1]. In particular, it contains a 1-complemented isometric copy of C_p for every $1 \leqslant p < \infty$. Now apply Corollary 3.3. $\square$

Corollary 3.5. *There are Banach spaces X without the AP for which $\mathscr{A}(X)$ is weakly amenable.*

Proof. If a Banach space X has the (AP) BAP, then every complemented subspace of X also has this property. Thus, if E is a Banach space without the (AP) BAP, the Banach space $X = E \oplus C_p$, $p = 0$ or $1 \leqslant p < \infty$, must fail the (AP) BAP, yet by Corollary 3.3, $\mathscr{A}(X)$ is weakly amenable. $\square$

With this we have shown that (AP) BAP is not a necessary condition for weak amenability of the algebra $\mathscr{A}(X)$.

As before, let $X = X_1 \oplus X_2$, but now let us suppose that both $\mathscr{A}(X_1)$ and $\mathscr{A}(X_2)$ are weakly amenable. Is $\mathscr{A}(X)$ weakly amenable?

We shall prove the following.

Theorem 3.6. *Suppose the hypotheses of Lemma* 3.1 *are satisfied. If* $\mathscr{A}(X_1)$ *and* $\mathscr{A}(X_2)$ *are weakly amenable, then* $\mathscr{A}(X)$ *is weakly amenable.*

Given a normed space X, $\kappa_X : X \to X''$ denotes the canonical embedding. For arbitrary normed spaces X and Y we denote by $X \hat{\otimes} Y$ (respectively, $X \check{\otimes} Y$) their projective (respectively, injective) tensor product.

Lemma 3.7. *Let X and Y be infinite-dimensional Banach spaces. If either X or Y has the BAP, then the bilinear form $b : \mathscr{F}(X, Y) \times \mathscr{F}(Y, X) \to \mathbb{C}$ defined by $b(R, S) := \mathrm{tr}(RS)$ $(R \in \mathscr{F}(X, Y),\ S \in \mathscr{F}(Y, X))$ is unbounded.*

Proof. Without loss of generality, let us suppose that X has the η-AP for some $\eta \geqslant 1$. Take $\delta, \epsilon > 0$. Let $S = \sum_{i=1}^n \lambda_i \otimes x_i \in Y' \otimes X$, and let $P \in \mathscr{F}(X)$ be such that $||P|| \leqslant \eta + \epsilon$ and $PS = S$ (we have used [DF, Section 16.9, Corollary]). By the Hahn–Banach Theorem and [DF, Section 3.2, Proposition (1)], there exists $T \in \mathscr{B}(Y', X') \cong (Y' \hat{\otimes} X)'$ with $||T|| = 1$ such that $||S||_\wedge = \mathrm{tr}(TS')$. Define $T_1 := P'T \in \mathscr{F}(Y', X')$. Then, by the principle of local reflexivity, there exists $T_2 : T_1'(X'') \to Y$ such that $||T_2|| \leqslant 1 + \delta$ and $(T_1'\kappa_X(x_i))(\lambda_i) = \lambda_i(T_2 T_1' \kappa_X(x_i))$ for all i. Take $R \in \mathscr{F}(X, Y)$ such that $R'' = \kappa_Y T_2 T_1'$ (it is not difficult to see that $R = T_2 T' \kappa_X P$). Then

$$\begin{aligned} ||S||_\wedge &= \mathrm{tr}(T_1 S') = \sum_{i=1}^n (T_1' \kappa_X(x_i))(\lambda_i) \\ &= \sum_{i=1}^n \lambda_i (T_2 T_1' \kappa_X(x_i)) = \sum_{i=1}^n (R'' \kappa_X(x_i))(\lambda_i) = \mathrm{tr}(RS). \end{aligned}$$

Since ϵ and δ can be arbitrarily small it follows that

$$||S||_\wedge \leqslant \sup\{|\mathrm{tr}(RS)| : R \in \mathscr{F}(X, Y),\ ||R|| \leqslant \eta\}.$$

Analogously, using the fact that $(Y' \check{\otimes} X)' \cong \mathscr{I}(Y', X')$ (see [DF, Section 10.1, Proposition]), it can be proved that

$$||S|| \leqslant \sup\{|\mathrm{tr}(RS)| : R \in \mathscr{F}(X, Y),\ ||R||_{\mathscr{I}} \leqslant \eta\}.$$

If b is bounded, then $||\,.\,||_{\mathscr{I}}$ and $||\,.\,||$ are equivalent in $\mathscr{F}(X, Y)$ and, consequently, $||\,.\,||_\wedge$ and $||\,.\,||$ are equivalent in $\mathscr{F}(Y, X)$, that is $Y' \hat{\otimes} X \simeq Y' \check{\otimes} X$. But since X has the BAP, either $\dim X < \infty$ or $\dim Y < \infty$ (see [Pi1]), which is a contradiction. Thus b must be unbounded. $\square$

Proof of Theorem 3.6. We show that condition (A*) of Theorem 2.1 is satisfied.

Let $T \in \Delta_X$, and let $W \in \mathscr{F}(X)$ be such that tr $W = 0$. By Lemma 3.1, it is sufficient to consider W with $W_{12} = 0$ and $W_{21} = 0$. Since $\mathscr{A}(X_1)$ and $\mathscr{A}(X_2)$ are weakly amenable, there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ and positive constants K_1, K_2 such that

$$|\mathrm{tr}((T_{11} - \lambda_1) W_1')| \leqslant K_1 ||W_1|| \quad (W_1 \in \mathscr{F}(X_1)) \tag{12}$$

and

$$|\mathrm{tr}((T_{22} - \lambda_2) W_2')| \leqslant K_2 ||W_2|| \quad (W_2 \in \mathscr{F}(X_2)). \tag{13}$$

We have

$$\mathrm{tr}(TW') = \mathrm{tr}((T_{11} - \lambda_1) W_{11}') + \mathrm{tr}((T_{22} - \lambda_2) W_{22}') \\ + \lambda_1 \,\mathrm{tr}(W_{11}) + \lambda_2 \,\mathrm{tr}(W_{22}).$$

Taking into account the fact that $\mathrm{tr}\, W_{11} = -\mathrm{tr}\, W_{22}$ and that tr is unbounded (recall that $\dim X_1 = \dim X_2 = \infty$), it is easily seen from the last identity together with (12) and (13) that (A^*) is satisfied if and only if $\lambda_1 = \lambda_2$.

Let $S^\sim \in \mathscr{F}(X_1, X_2)$ and $R^\sim \in \mathscr{F}(X_2, X_1)$. Define $R := \imath_1 R^\sim \gamma_2$ and $S := \imath_2 S^\sim \gamma_1$. Then

$$\begin{aligned} b_T(R,S) &= \mathrm{tr}(T_{11}(R^\sim S^\sim)') - \mathrm{tr}(T_{22}(S^\sim R^\sim)') \\ &= \mathrm{tr}((T_{11} - \lambda_1)(R^\sim S^\sim)') - \mathrm{tr}((T_{22} - \lambda_2)(S^\sim R^\sim)') \\ &\quad + (\lambda_1 - \lambda_2)\,\mathrm{tr}(R^\sim S^\sim). \end{aligned}$$

It follows that

$$|\lambda_1 - \lambda_2||\mathrm{tr}(R^\sim S^\sim)| \leqslant (||b_T|| + K_1 + K_2)||R^\sim||\,||S^\sim|| \\ (S^\sim \in \mathscr{F}(X_1, X_2),\ R^\sim \in \mathscr{F}(X_2, X_1)). \tag{14}$$

Since by hypothesis either X_1 or X_2 has the BAP (see Lemma 3.1), the bilinear form of Lemma 3.7 cannot be bounded. Thus, (14) is possible if and only if $\lambda_1 = \lambda_2$, and so (A^*) is satisfied. $\square$

Corollary 3.8. *Let X_i be an $\mathscr{L}_{p_i}$-space, $1 \leqslant p_i \leqslant \infty$ $(1 \leqslant i \leqslant n)$. Then $\mathscr{A}(X_1 \oplus X_2 \oplus \cdots \oplus X_n)$ is weakly amenable.*

Proof. For $n = 2$ the result is an immediate consequence of Theorem 3.6. The general case follows by induction. $\square$

Remark. Note that $\mathscr{A}(l_p \oplus l_q)$ $(1 < p, q < \infty)$ is amenable if and only if $p = q$ or $p = 2$ or $q = 2$. It is also known that $\mathscr{A}(c_0 \oplus l_p)$ is not amenable for $1 < p < 2$ and that $\mathscr{A}(l_1 \oplus l_p)$ is not amenable for $2 < p < \infty$. All these results can be found in [GJW, Theorem 6.9].

Corollary 3.9. *Let E be a Banach space with the BAP. If $\mathscr{A}(E)$ is weakly amenable, then $\mathscr{A}(l_p^m(E))$ $(1 \leqslant p \leqslant \infty, m \in \mathbb{N})$ is weakly amenable.*

Proof. Since E has the BAP, the same holds for $E \oplus E$. Thus, by Theorem 3.6, $\mathscr{A}(E \oplus E)$ is weakly amenable. Now continue by induction in the obvious way. $\square$

Remark. The converse of the above corollary is true for every Banach space and not only for those that satisfy BAP. This follows easily from Theorem 2.1. In fact, it is not hard to see that, if $T \in \Delta_E$, then $\mathscr{T} : l_p^m(E) \to l_p^m(E)$ defined by $\mathscr{T}((x_i)_{i=1}^m) := (Tx_i)_{i=1}^m$ $((x_i)_{i=1}^m \in l_p^m(E))$ belongs to $\Delta_{l_p^m(E)}$. Since $\mathscr{A}(l_p^m(E))$ is weakly amenable, there are $\lambda \in \mathbb{C}$ and $K > 0$ such that

$$|\mathrm{tr}((\mathscr{T} - \lambda)\mathscr{W}')| \leqslant K||\mathscr{W}|| \quad (\mathscr{W} \in \mathscr{F}(l_p^m(E))).$$

For each $W \in \mathscr{F}(E)$ define $\mathscr{W} \in \mathscr{F}(l_p^m(E))$ by

$$\mathscr{W}((x_i)_{i=1}^m) := (Wx_1, 0, \ldots, 0) \quad ((x_i)_{i=1}^m \in l_p^m(E)).$$

Then $\mathrm{tr}\, W = \mathrm{tr}\, \mathscr{W}$, $\mathrm{tr}(TW') = \mathrm{tr}(\mathscr{T}\mathscr{W}')$ and $||W|| = ||\mathscr{W}||$. The rest is clear.

Let us now turn to the study of the relationship between the weak amenability of $\mathscr{A}(X)$ and the weak amenability of $\mathscr{A}(X')$. We have the following result analogous to Corollary 5.3 of [GJW].

Proposition 3.10. *Let X be a Banach space. If $\mathscr{A}(X')$ is weakly amenable, then $\mathscr{A}(X)$ is weakly amenable.*

Proof. Let $T \in \Delta_X$. Let $R^\sim = \sum_{j=1}^n y_j'' \otimes \tau_j$ and $S^\sim = \sum_{i=1}^m x_i'' \otimes \lambda_i$ be arbitrary elements in $\mathscr{F}(X')$. By the principle of local reflexivity there exists

$$T_1 : \mathrm{sp}\{x_i'', y_j'' : 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\} \to X$$

such that $||T_1|| \leqslant 2$ and $\xi(f) = f(T_1\xi)$ $(\xi \in \mathrm{sp}\{x_i'', y_j'' : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}, f \in \mathrm{sp}\ \{\lambda_i, \tau_j, T\lambda_i, T\tau_j : 1 \leqslant i \leqslant m,\quad 1 \leqslant j \leqslant n\})$. Let $R = \sum_{j=1}^n \tau_j \otimes T_1 y_j''$ and $S = \sum_{i=1}^m \lambda_i \otimes T_1 x_i''$. Then $S'' = \kappa_X\, T_1(S^\sim)'$, $R'' = \kappa_X\, T_1(R^\sim)'$, and

$$\begin{aligned}|\mathrm{tr}(T'(R^\sim S^\sim - S^\sim R^\sim)')| &= |\mathrm{tr}(T(SR - RS)')| \leqslant ||b_T||\, ||R||\, ||S|| \\ &\leqslant 4||b_T||\, ||R^\sim||\, ||S^\sim||,\end{aligned}$$

that is, $T' \in \Delta_{X'}$.

Since $\mathscr{A}(X')$ is weakly amenable, by Theorem 2.1, there exist $\lambda \in \mathbb{C}$ and a positive constant K such that

$$|\mathrm{tr}((T' - \lambda)\tilde{W}')| \leqslant K||\tilde{W}|| \quad (\tilde{W} \in \mathscr{F}(X')).$$

Thus

$$|\mathrm{tr}((T-\lambda)W')| = |\mathrm{tr}((T'-\lambda)W'')| \leqslant K\|W\| \qquad (W \in \mathscr{F}(X)),$$

proving that $\mathscr{A}(X)$ is weakly amenable. □

Let E be a Banach space, and let K be a compact topological space. We write $C(K,E)$ for the Banach space of continuous, E-valued functions on K, with the uniform norm.

Corollary 3.11. *Let E and K be as above. Suppose that* $\dim C(K) = \infty$. *If E' has the Radon–Nikodým property (RNP) and E'' has the BAP then $\mathscr{A}(C(K,E))$ is weakly amenable.*

Proof. The dual of $C(K)$ is isometrically isomorphic to an infinite-dimensional Banach space of the form $L_1(\mu)$ for some measure μ, and it is well-known that $L_1(\mu)\hat{\otimes}F \cong L_1(\mu,F)$ for every Banach space F. Thus, since E' has the RNP and $C(K)$ has the AP, we have from [DF, Section 4.2, Example 2 and Theorem 16.6] that

$$\begin{aligned} C(K,E)' &\cong (C(K)\check{\otimes}E)' \cong C(K)'\hat{\otimes}E' \\ &\cong L_1(\mu)\hat{\otimes}E' \cong L_1(\mu,E'). \end{aligned}$$

By [Bl, Theorem 4.1], $\mathscr{A}(L_1(\mu,E'))$ is weakly amenable, and now the desired result follows by Proposition 3.10. □

It is natural now to ask about the converse of Proposition 3.10, that is: does the weak amenability of $\mathscr{A}(X)$ imply the weak amenability of $\mathscr{A}(X')$? We shall see in the next proposition that under certain additional hypotheses on X, or, more precisely, on X'', this does in fact hold.

Proposition 3.12. *Let X be a Banach space such that $\mathscr{A}(X)$ is weakly amenable. If $\kappa_X(X)$ is complemented in X'' (in particular, if X is a dual space), and if $|W|_X \leqslant \rho\|W\|$ $(W\in\mathscr{F}(X''))$ for some constant $\rho>0$ (in particular, if X'' has the BAP), then $\mathscr{A}(X'')$ (and by Proposition 3.10, $\mathscr{A}(X')$ too) is weakly amenable.*

Proof. It is well known that $\kappa_X(X)$ is complemented in X'' whenever X is a dual space, and it is shown in [BDG, Proposition 4(a)(2)] that, if X'' has the BAP, then there exists a constant $\rho>0$ such that $|W|_X \leqslant \rho\|W\|$ $(W\in\mathscr{F}(X''))$. The rest is clearly a consequence of Proposition 3.2. □

Using the last proposition we recover the following (see Corollary 3.4).

Corollary 3.13. *$\mathscr{A}(C_\infty)$ is weakly amenable.*

Proof. The space C_1 $(= C_0')$ satisfies the hypotheses of the last proposition, and $C_\infty = C_1'$. $\square$

4. A necessary condition

We have seen in the preceding section that BAP is not a necessary condition for the weak amenability of $\mathscr{A}(X)$. Our next proposition gives a necessary condition for the weak amenability of this algebra. Note that the argument is similar to that used in Proposition 3.6.

Proposition 4.1 (Necessary condition). *Let X_1 and X_2 be infinite-dimensional Banach spaces, and let $X = X_1 \oplus X_2$. If the bilinear map $b : \mathscr{F}(X_1, X_2) \times \mathscr{F}(X_2, X_1) \to \mathbb{C}$ defined by $b(S, R) := \mathrm{tr}(SR)$ is bounded, then $\mathscr{A}(X)$ is not weakly amenable.*

Proof. Take $\lambda_1, \lambda_2 \in \mathbb{C}$ distinct, and let $T \in \mathscr{B}(X')$ be defined by $T_{ij} := 0$ if $i \neq j$ and $T_{ii} := \lambda_i I_{X_i}$ $(1 \leqslant i, j \leqslant 2)$ (see the notation at the beginning of Section 3). Supposing the bilinear map b is bounded, we have

$$|\mathrm{tr}(T(RS - SR)')| = |(\lambda_1 - \lambda_2)\mathrm{tr}(R_{12}S_{21}) - (\lambda_1 - \lambda_2)\mathrm{tr}(S_{12}R_{21})|$$
$$\leqslant 2\,|\lambda_1 - \lambda_2| \|b\| \, \|R\| \, \|S\| \quad (R, S \in \mathscr{F}(X)).$$

On the other hand, since $\dim X_i = \infty$, $\mathrm{tr} : \mathscr{F}(X_i) \to \mathbb{C}$ $(i = 1, 2)$ is unbounded, and so condition (A*) of Theorem 2.1 cannot be satisfied. To see this, take $W \in \mathscr{F}(X)$ with $\mathrm{tr}\, W = 0$. Then we have

$$\mathrm{tr}(TW') = (\lambda_1 - \lambda_2)\mathrm{tr}\, W_{11} = (\lambda_2 - \lambda_1)\mathrm{tr}\, W_{22}.$$

For any positive integer n, there exists $W_i \in \mathscr{F}(X_i)$ such that $|\mathrm{tr}\, W_i| > n\|W_i\|$ $(i = 1, 2)$. Clearly, we may always suppose that $\mathrm{tr}\, W_1 = -\mathrm{tr}\, W_2$. Let $W := \imath_1 W_1 \gamma_1 + \imath_2 W_2 \gamma_2$. Then

$$n\|W\| \leqslant n(\|W_1\| + \|W_2\|) < \frac{2}{|\lambda_1 - \lambda_2|}|\mathrm{tr}(TW')|.$$

Thus, as we claimed, (A*) is not satisfied, and consequently, by Theorem 2.1, $\mathscr{A}(X)$ is not weakly amenable. $\square$

The last proposition essentially says that, given a Banach space X, for $\mathscr{A}(X)$ to be weakly amenable it is necessary that whenever P is a continuous projection on X such that both $\mathrm{rg}\, P$ and $\ker P$ are infinite-dimensional, the bilinear map $b_P : \mathscr{F}(\ker P, \mathrm{rg}\, P)) \times \mathscr{F}(\mathrm{rg}\, P, \ker P) \to \mathbb{C}$ defined by $b_P(S, R) := \mathrm{tr}(SR)$ is unbounded (compare this with the definition of an *approximately primary* Banach space given in [GJW]).

Corollary 4.2. *Let X and Y be infinite-dimensional Banach spaces such that $X \check{\otimes} Y = X \hat{\otimes} Y$ holds isomorphically. Then $\mathscr{A}(X \oplus Y')$ is not weakly amenable.*

Proof. We can identify $\mathscr{F}(X, Y')$ with a linear subspace of $(X \check{\otimes} Y)'$ by associating to each $V \in \mathscr{F}(X, Y')$ the linear functional $\varphi_V : X \check{\otimes} Y \to \mathbb{C}$ defined by

$$\varphi_V(x \otimes y) := (Vx)(y) \quad (x \in X, y \in Y).$$

Since $X \check{\otimes} Y \simeq X \hat{\otimes} Y$, we have $(X \check{\otimes} Y)' \simeq (X \hat{\otimes} Y)' \cong \mathscr{B}(X, Y')$, and so there exists a constant $C > 0$ such that $\|\varphi_V\| \leqslant C\|V\|$ $(V \in X' \otimes Y' = \mathscr{F}(X, Y'))$.

Let $R \in \mathscr{F}(X, Y') = X' \otimes Y'$ fixed. By [DF, Theorem 6.7 (Extension Lemma)], φ_R has a natural extension $\varphi_R^{\wedge} \in (X \check{\otimes} Y'')'$ defined by

$$\langle \varphi_R^{\wedge}, x \otimes y'' \rangle = \langle y'', Rx \rangle = \mathrm{tr}(R\,(y'' \otimes x)) \quad (x \in X, y'' \in Y'')$$

such that $\|\varphi_R^{\wedge}\| = \|\varphi_R\|$. It follows that

$$\begin{aligned} |\mathrm{tr}(RS)| &= |\langle \varphi_R^{\wedge}, S \rangle| \leqslant \|\varphi_R^{\wedge}\|\, \|S\| \\ &\leqslant C\|R\|\, \|S\| \quad (R \in \mathscr{F}(X, Y'), S \in \mathscr{F}(Y', X)). \end{aligned}$$

Now, to finish our proof we just need to apply Proposition 4.1. $\square$

Before passing to our next result we need to recall some terminology. Let $\Upsilon := \{-1, 1\}^{\mathbb{N}}$, and let μ be the normalized Haar measure on Υ, that is, the infinite product of $\frac{1}{2}(\delta_1 + \delta_{-1})$. Denote by $\varepsilon_n : \Upsilon \to \{-1, 1\}$ the nth coordinate projection on Υ. A Banach space X is said to have *cotype* q, $2 \leqslant q < \infty$, if there is a constant C such that, for all finite subsets $\{x_1, x_2, \ldots, x_n\}$ of X, we have

$$\left(\sum_{i=1}^{n} \|x_i\|^q \right)^{\frac{1}{q}} \leqslant C \int_{\Upsilon} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| d\mu.$$

We denote by $C_q(X)$ the smallest constant C such that the above inequality holds.

Recall also that an operator T from a Banach space X into a Banach space Y is said to be *p-summing*, $1 \leqslant p < \infty$, if there is a constant C such that, for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of X, we have

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p \right)^{\frac{1}{p}} \leqslant C \sup \left\{ \left(\sum_{i=1}^{n} |\xi(x_i)|^p \right)^{\frac{1}{p}} : \xi \in X', \ \|\xi\| \leqslant 1 \right\}.$$

The smallest constant C for which the last inequality holds is called the p-summing norm of T and is denoted by $\pi_p(T)$. The p-summing operators form a linear space,

$\Pi_p(X, Y)$, and π_p is a norm on it. Moreover, $\Pi_p(X, Y)$, with the norm π_p is a Banach space. Trivially, $\|T\| \leqslant \pi_p(T)$ $(T \in \Pi_p(X, Y))$ so that $\Pi_p(X, Y) \subset \mathscr{B}(X, Y)$.

It is easily seen that, if $T \in \Pi_p(X, Y)$ $(1 \leqslant p < \infty)$ and $V : Y \to Y_1$ and $W : X_1 \to X$ are bounded linear operators between Banach spaces, then we have the "ideal property": $\pi_p(VTW) \leqslant \|V\| \pi_p(T) \|W\|$.

A Banach space X is said to verify *Grothendieck's theorem* or to be a *G.T. space* if $\mathscr{B}(X, l_2) = \Pi_1(X, l_2)$.

Corollary 4.3. *Let X be a Banach space of cotype 2 whose dual X' is a G.T. space of cotype 2. Then $\mathscr{A}(X \oplus X)$ is not weakly amenable.*

Proof. For any pair of operators $R, S \in \mathscr{F}(X)$ we have, by [Ja, Propositions 4.2 and 17.7] and [Pi5, Lemma 3], that

$$\begin{aligned} |\mathrm{tr}(RS)| &\leqslant \|RS\|_{\wedge} \leqslant \pi_2(R)\pi_2(S) \\ &\leqslant K^2 \|R\| \, \|S\|. \end{aligned}$$

The desired result follows from the last inequality and Proposition 4.1. $\square$

That there are infinite-dimensional Banach spaces satisfying the hypotheses of Corollaries 4.2 and 4.3 is a famous result of Pisier [Pi3, Theorem 3.2]. In fact, Pisier proved that every Banach space of cotype 2 can be embedded isometrically into a Banach space P satisfying the two following conditions:

(p.1) $P \hat{\otimes} P = P \check{\otimes} P$ algebraically and topologically; and
(p.2) P and P' are both G.T. spaces of cotype 2.

We shall refer to any infinite-dimensional Banach space satisfying both conditions above as a *Pisier space*. Note that a Pisier space necessarily fails to have the AP (see [Pi1]).

Corollary 4.4. *Let P be a Pisier space. Then $\mathscr{A}(P \oplus P')$ and $\mathscr{A}(P \oplus P)$ are not weakly amenable.*

Proof. It follows from the definition of a Pisier space together with Corollaries 4.2 and 4.3, respectively. $\square$

Remark. Note that if P is any Pisier space, then so is $P \oplus P$. Thus, by the last corollary, there are Pisier spaces for which the algebra of approximable operators is not weakly amenable.

Remark. It is implicit in the hypotheses of Proposition 3.2 that the subspace X_1 of X should be topologically complemented in X. At first glance it is difficult to get a real feeling about the importance of this assumption. Our last result, together with Corollaries 4.2 and 4.3, emphasizes its significance. In fact, since l_2 has cotype 2, there is a Pisier space P that contains l_2 isometrically. Then, though (i) l_2' has the BAP, (ii) $\mathscr{A}(l_2)$ is weakly amenable, and (iii) by [Pi4, Theorem 4.1] there exists a constant C such that every operator U in $\mathscr{F}(P\oplus P)$ (in $\mathscr{F}(P\oplus P')$) satisfies $|U|_{l_2} \leqslant C||U||$ (see (8)); the algebra $\mathscr{A}(P\oplus P)$ (resp. $\mathscr{A}(P\oplus P')$) is not weakly amenable. What fails in these examples is that $P\oplus P$ and $P\oplus P'$ cannot contain a *complemented* isomorphic copy of l_2, for $P\oplus P$ and $P\oplus P'$ satisfy (p.2), and by [Pi3, Corollary 3.5] there exists a constant $\delta > 0$ such that for any finite-rank projection Q we have $||Q|| \geqslant \delta\sqrt{\text{rank } Q}$. We will see, however, in Lemma 4.5, that $\mathscr{A}(P\oplus P\oplus l_2)$ and $\mathscr{A}(P\oplus P'\oplus l_2)$ are weakly amenable.

Another necessary condition for the weak amenability of $\mathscr{A}(X)$ has been given by Grønbæk. He noted that $\mathscr{A}(X)$ is weakly amenable only if the product map $\pi : \mathscr{A}(X)\hat{\otimes}\mathscr{A}(X) \to \mathscr{A}(X)$ is onto. The latter is easily seen to be equivalent to the following condition:

(P) There is a constant C such that for every $U \in \mathscr{A}(X)$, there are sequences (V_n) and (W_n) in $\mathscr{A}(X)$ such that $U = \sum_{n=1}^{\infty} V_n W_n$ and $\sum_{n=1}^{\infty} ||V_n||\,||W_n|| \leqslant C||U||$.

In [Pi5] Pisier constructed a Banach space X that fails the above property (and hence such that $\mathscr{A}(X)$ is not weakly amenable). This was the first known example of a Banach space X for which $\mathscr{A}(X)$ was not weakly amenable. We should point out here that our necessary condition is not equivalent to Grønbæk's. In fact, if P is a Pisier space, then it is clear from Corollary 4.3 that $P\oplus P$ does not satisfy our necessary condition. However, it is easily verified that $P\oplus P$ satisfies (P). This not only shows that our necessary condition is not equivalent to Grønbæk's, but also shows that it is no worse than the latter. Moreover, the examples we have provided in this section are essentially different from the one given in [Pi5].

We end this section with a result on infinite direct sums. It is in some sense a continuation of the results of Section 3. As we saw there for a Banach space X with BAP, the algebra $\mathscr{A}(l_p^m(X))$ ($1 \leqslant p \leqslant \infty$, $m \in \mathbb{N}$) is weakly amenable if and only if $\mathscr{A}(X)$ is weakly amenable. It will be shown here that the situation is significantly improved if we consider infinite sums instead. In fact, we will see in Proposition 4.6 below that, starting with a Banach space X such that $\mathscr{A}(X)$ is not weakly amenable and summing up infinitely many copies of X (in the appropriate sense), we might end up with a Banach space for which the algebra of approximable operators is weakly amenable.

We shall need the following lemma, which has independent interest. It is in fact a corollary of Proposition 3.2.

Lemma 4.5. *Let X be a Banach space of cotype 2 with dual X' of cotype 2 also, (in particular $X = P\oplus P$ or $X = P\oplus P'$ with P any Pisier space). Then for any $1 < p < \infty$, $\mathscr{A}(X\oplus l_p)$ is weakly amenable.*

Proof. Let $U \in \mathscr{F}(X)$. Since both X and X' have cotype 2, it can be shown using [Pi4, Theorem 4.1] that there exist a positive integer $\varkappa$ and operators $U_1 \in \mathscr{F}(X, l_2^{\varkappa})$ and $U_2 \in \mathscr{F}(l_2^{\varkappa}, X)$ such that $U = U_2 U_1$ and $||U_1||\,||U_2|| \leqslant (C_X + 1)||U||$ $(C_X = (2C_2(X')C_2(X))^{\frac{3}{2}})$.

It is known (see for instance [W, III.A. Exercise 6]) that l_p $(1<p<\infty)$ contains uniformly complemented l_2^n's, that is, that there are constants $M_1 \geqslant 1$ and $M_2 \geqslant 1$ such that for every $n \in \mathbb{N}$ there exists an n-dimensional subspace $E_n \subset l_p$ and a continuous projection $P_n : l_p \to E_n$ satisfying: $d(E_n, l_2^n) \leqslant M_1$ and $||P_n|| \leqslant M_2$. Take $T : E_\varkappa \to l_2^\varkappa$ such that $||T||\,||T^{-1}|| \leqslant M_1$ and let $\iota_\varkappa : E_\varkappa \to l_p$ denote the inclusion map. Define $R := U_2 T P_\varkappa$ and $S := \iota_\varkappa T^{-1} U_1$. Then $U = RS$ and $|U|_{l_p} \leqslant ||R||\,||S|| \leqslant \rho ||U||$ (see (8)), where $\rho = M_1 M_2 (C_X + 1)$.

With this we have shown that $|U|_{l_p} \leqslant \rho\, ||U||$ $(U \in \mathscr{F}(X))$ for some constant ρ independent of U. Since l_p has the BAP and $\mathscr{A}(l_p)$ is weakly amenable, we can apply Proposition 3.2 to conclude that $\mathscr{A}(X \oplus l_p)$ is weakly amenable. $\square$

Remark. Pisier proved (see [Pi2]) that any Banach space not containing l_1^n's uniformly must contain uniformly complemented l_2^n's. Thus, it is clear from the preceding proof that we can replace l_p, in the hypotheses of Corollary 4.5, with any Banach space X having the BAP, not containing l_1^n's uniformly, and such that $\mathscr{A}(X)$ is weakly amenable.

Proposition 4.6. *Let X be an infinite dimensional Banach space of cotype 2 with dual X' of cotype 2. Then $\mathscr{A}(l_2(X))$ is weakly amenable. In particular $\mathscr{A}(l_2(P))$ and $\mathscr{A}(l_2(P \oplus P'))$ are weakly amenable for any Pisier space P.*

Proof. It is easy to check that both $l_2(X)$ and $(l_2(X))'$ $(= l_2(X'))$ have cotype 2. Since $l_2(X) \simeq l_2(X) \oplus l_2$, to finish our proof we just need to apply Corollary 4.5. $\square$

Note that $l_2(P)$ and $l_2(P \oplus P')$ (where P is any Pisier space) also fail AP.

5. Insufficiency of the bounded approximation property

We have already seen that the BAP is not necessary for the weak amenability of the algebra of approximable operators on a Banach space (Corollary 3.5). In this section we consider the question of whether or not BAP is sufficient. We will see that there are Banach spaces X with the BAP such that $\mathscr{A}(X)$ is not weakly amenable.

We first construct a reflexive Banach space E with unconditional basis such that $\mathscr{A}(E)$ is not weakly amenable (Proposition 5.3). Then we will use a refinement of the former construction to show that there are indeed infinitely many non-isomorphic Banach spaces with the same properties. We have preferred to organize things in this way so as not to complicate the main construction with details that are irrelevant to the (non)weak amenability of the algebra $\mathscr{A}(E)$.

Our main example is based on the following result of Tomczak-Jaegermann.

Lemma 5.1 (Tomczak-Jaegermann). *Let (ϵ_n) be a sequence of positive numbers, and let $(p_n) \subset]1,2[$ and $(k_n) \subset \mathbb{N}$ be strictly increasing sequences satisfying the* *inequalities*

$$k_n^{\frac{1}{p_n}-\frac{1}{2}} \geqslant \epsilon_n^{-1} \quad \textit{and} \quad k_n^{\frac{1}{p_{n+1}}-\frac{1}{2}} \leqslant 2, \tag{15}$$

Then the following holds:

1. $\sum_{i=1}^{n} k_i \geqslant \sum_{i=1}^{n} \epsilon_i^{-1}$ $(n \in \mathbb{N})$; *and*
2. *there exists a positive constant* C, *such that for every* $n \in \mathbb{N}$, *if* $S : l_{p_n}^{k_n} \to (\sum_{i=n+1}^{\infty} l_{p_i}^{k_i})_{l_2}$ and $R : (\sum_{i=n+1}^{\infty} l_{p_i}^{k_i})_{l_2} \to l_{p_n}^{k_n}$ *are bounded operators*, *then*

$$|\mathrm{tr}(RS)| \leqslant C\epsilon_n k_n ||R||\, ||S||.$$

We need the following auxiliary result.

Lemma 5.2. *Let* $U : l_p^k \to l_2^{k'}$ and $V : l_2^{k'} \to l_p^k$ $(1 \leqslant p \leqslant 2)$ *be linear operators. Then there exists a constant* $\tilde{C}$ *independent of* p, k *and* k' *such that*

$$|\mathrm{tr}(VU)| \leqslant \tilde{C} k^{\frac{3}{2}-\frac{1}{p}} ||V||\, ||U||.$$

Proof. Let U and V be as in the statement of the lemma. By [Ja, Propositions 4.2 and 17.7] and [Pi4, Theorem 1.11], we have

$$|\mathrm{tr}(VU)| \leqslant \pi_2(V)\pi_2(U) \leqslant \sqrt{k}||V||\pi_2(U).$$

Let $I_{p,1} : l_p^k \to l_1^k$ and $I_{1,p} : l_1^k \to l_p^k$ be the formal identity operators, so that $||I_{p,1}|| = k^{1-\frac{1}{p}}$ and $||I_{1,p}|| = 1$. Then, by the ideal property of $\Pi_2(l_1^k, l_2^{k'})$ and the Little Grothendieck Theorem ([Pi4, Theorem 5.10]), we have

$$\pi_2(U) \leqslant \pi_2(U\, I_{1,p})||I_{p,1}|| \leqslant C_1 k^{1-\frac{1}{p}} ||U\, I_{1,p}|| \leqslant C_1 k^{1-\frac{1}{p}} ||U||.$$

Combining both estimates above, we obtain the desired result with $\tilde{C} = C_1$. □

Proof of Lemma 5.1. Note that from (15), 1 is automatically satisfied, so we only need to show 2.

Fix $n \in \mathbb{N}$ and let $S : l_{p_n}^{k_n} \to (\sum_{i=n+1}^{\infty} l_{p_i}^{k_i})_{l_2}$ and $R : (\sum_{i=n+1}^{\infty} l_{p_i}^{k_i})_{l_2} \to l_{p_n}^{k_n}$ be bounded operators. First we show that $d(S(l_{p_n}^{k_n}), l_2^{k'}) \leqslant 2$ (where $k' = \dim S(l_{p_n}^{k_n})$). For this, let $F_i := \gamma_i(S(l_{p_n}^{k_n}))$ $(i = n+1, \ldots)$, where $\gamma_i : (\sum_{j=n+1}^{\infty} l_{p_j}^{k_j})_{l_2} \to l_{p_i}^{k_i}$ is the canonical i-th coordinate projection. By a result of Lewis (see [Le]) and our choice of (p_n) and (k_n)

we have

$$d(F_i, l_2^{\dim F_i}) \leqslant \dim F_i^{\left|\frac{1}{p_i}-\frac{1}{2}\right|} \leqslant k_n^{\frac{1}{p_i}-\frac{1}{2}}$$

$$\leqslant k_n^{\frac{1}{p_{n+1}}-\frac{1}{2}} \leqslant 2.$$

Thus the space $(\sum_{i=n+1}^{\infty} F_i)_{l_2}$ is 2-isomorphic to a Hilbert space and since $S(l_{p_n}^{k_n})$ is a finite-dimensional subspace of $(\sum_{i=n+1}^{\infty} F_i)_{l_2}$ the same must be true for $S(l_{p_n}^{k_n})$.

Now let $\tilde{S}$ and $\tilde{R}$ denote the corestriction of S to $S(l_{p_n}^{k_n})$ and the restriction of R to $S(l_{p_n}^{k_n})$, respectively. Let $T : S(l_{p_n}^{k_n}) \to l_2^{k'}$ be a linear isomorphism such that $||T||\,||T^{-1}|| \leqslant 2$ (by our previous result such an isomorphism exists). Clearly, $RSx = \tilde{R}T^{-1}T\tilde{S}x$ $(x \in l_{p_n}^{k_n})$, and so, by Lemma 5.2, we have

$$|\mathrm{tr}(RS)| = |\mathrm{tr}(\tilde{R}T^{-1}T\tilde{S})| \leqslant \tilde{C}k_n^{\frac{3}{2}-\frac{1}{p_n}}||\tilde{R}T^{-1}||\,||T\tilde{S}||$$

$$\leqslant 2\tilde{C}k_n\epsilon_n||R||\,||S||,$$

that is, condition 2 is satisfied. □

The proof of Lemma 5.1 that we have presented here is also due to Tomczak-Jaegermann.

Proposition 5.3. *Let (ϵ_n) be a sequence of positive numbers such that $\sum_n \epsilon_n < \infty$, and let $(p_n) \subset \,]1,2[$ and $(k_n) \subset \mathbb{N}$ be strictly increasing sequences satisfying inequalities* (15). *Then $\mathscr{A}(X)$ is not weakly amenable for $X = (\sum \oplus_n l_{p_n}^{k_n})_2$.*

Proof. Denote by γ_i (respectively, ι_i) the canonical ith coordinate projection (respectively, embedding) with respect to X. Then define $P_n := \sum_{i=1}^{n} \iota_i\gamma_i$ $(n \in \mathbb{N})$, and let $\mathfrak{A}$ be the algebra of those operators $W \in \mathscr{F}(X)$ such that $W = P_nWP_n$ for some $n \in \mathbb{N}$. It is easily seen that $\mathfrak{A}$ is dense in $\mathscr{F}(X)$ in the projective norm.

Let $T := \sum_n k_n^{-1}\iota_n\gamma_n$. We show that T belongs to Δ_X and $\lim_n |\mathrm{tr}((T-\lambda)P_n)| = \infty$ $(\lambda \in \mathbb{C})$. Then, since $||P_n|| = 1$ $(n \in \mathbb{N})$, it follows by Corollary 2.4 that $\mathscr{A}(X)$ cannot be weakly amenable.

Let $R, S \in \mathfrak{A}$. Then $S = P_nSP_n$ and $R = P_nRP_n$ for some $n \in \mathbb{N}$. To simplify our notation, in what follows S_{ij} (respectively, R_{ij}) will denote the operator $\iota_i\gamma_iS\iota_j\gamma_j$ (respectively, $\iota_i\gamma_iR\iota_j\gamma_j$) $(1 \leqslant i,j \leqslant n)$.

Let $\vartheta_i := k_i^{-1}$ $(i \in \mathbb{N})$. We have the following:

$$\begin{aligned}
|b_T(R,S)| &= \left| \sum_{i=1}^{n} \vartheta_i \operatorname{tr}\left(\sum_{j=1}^{n} (R_{ij}S_{ji} - S_{ij}R_{ji}) \right) \right| \\
&= \left| \sum_{1 \leqslant i,j \leqslant n} \vartheta_i \operatorname{tr}(R_{ij}S_{ji}) - \sum_{1 \leqslant i,j \leqslant n} \vartheta_j \operatorname{tr}(R_{ij}S_{ji}) \right| \\
&= \left| \sum_{i=1}^{n-1} \sum_{i<j \leqslant n} (\vartheta_i - \vartheta_j)\operatorname{tr}(R_{ij}S_{ji}) \right. \\
&\quad \left. + \sum_{i=2}^{n} \sum_{1 \leqslant j < i} (\vartheta_i - \vartheta_j)\operatorname{tr}(R_{ij}S_{ji}) \right| \\
&= \sum_{i=1}^{n-1} \left| \sum_{i<j \leqslant n} (\vartheta_i - \vartheta_j)\operatorname{tr}(R_{ij}S_{ji}) \right| \\
&\quad + \sum_{j=1}^{n-1} \left| \sum_{j<i \leqslant n} (\vartheta_i - \vartheta_j)\operatorname{tr}(R_{ij}S_{ji}) \right|.
\end{aligned}$$

Since $R_{ij}S_{li} = 0$ whenever $j \neq l$, the last expression can be written as

$$\begin{aligned}
&\sum_{i=1}^{n-1} \vartheta_i \left| \operatorname{tr}\left(\left(\sum_{i<j \leqslant n} R_{ij} \right) \left(\sum_{i<l \leqslant n} \frac{\vartheta_i - \vartheta_l}{\vartheta_i} S_{li} \right) \right) \right| \\
&\quad + \sum_{j=1}^{n-1} \vartheta_j \left| \operatorname{tr}\left(\left(\sum_{j<i \leqslant n} S_{ji} \right) \left(\sum_{j<l \leqslant n} \frac{\vartheta_j - \vartheta_l}{\vartheta_j} R_{lj} \right) \right) \right|.
\end{aligned}$$

Let

$$\begin{aligned}
R_i &= \sum_{i<j \leqslant n} R_{ij}, \quad S^i = \sum_{i<j \leqslant n} \frac{\vartheta_i - \vartheta_j}{\vartheta_i} S_{ji}; \quad S_j = \sum_{j<i \leqslant n} S_{ji}, \\
R^j &= \sum_{j<i \leqslant n} \frac{\vartheta_j - \vartheta_i}{\vartheta_j} R_{ij} \quad \text{and} \quad \Theta_i = \sum_{k=i+1}^{n} \frac{\vartheta_i - \vartheta_k}{\vartheta_i} \iota_k \gamma_k \quad (1 \leqslant i,j \leqslant n).
\end{aligned}$$

In particular: $R_i x = \iota_i \gamma_i R(I - P_i)x$ and $S^i x = \Theta_i (I - P_i) S \iota_i \gamma_i x$ $(x \in X)$, and it follows that $||R_i|| = ||R||$ and $||S^i|| \leqslant ||S||$ $(1 \leqslant i \leqslant n)$. Then we have shown that

$$|b_T(R,S)| \leqslant \sum_{i=1}^{n} \vartheta_i |\operatorname{tr}(R_i S^i)| + \sum_{j=1}^{n} \vartheta_j |\operatorname{tr}(S_j R^j)|. \tag{16}$$

Let $\tilde{S^i}$ denote the corestriction of S^i to $(I - P_i)X$, and let $\tilde{R}_i$ denote the restriction of R_i to $(I - P_i)X$, so that $||S^i|| = ||\tilde{S^i}||$ and $||R_i|| \geqslant ||\tilde{R}_i||$. Then, since $(I - P_i)X$ is isometrically isomorphic to $(\sum \oplus_{j=i+1}^{\infty} l_{p_j}^{k_j})_2$, we have by Lemma 5.1 that

$$\begin{aligned}|\mathrm{tr}(R_i S^i)| &= |\mathrm{tr}(\gamma_i R_i S^i \iota_i)| = |\mathrm{tr}(\gamma_i \tilde{R}_i \tilde{S}^i \iota_i)| \\ &\leqslant C\epsilon_i k_i ||\gamma_i \tilde{R}_i|| ||\tilde{S}^i \iota_i|| \\ &\leqslant C\epsilon_i k_i ||R_i|| \, ||S^i|| \\ &\leqslant C\epsilon_i k_i ||R|| \, ||S|| \quad (1 \leqslant i \leqslant n). \end{aligned} \tag{17}$$

In a similar way, we obtain that

$$|\mathrm{tr}(S_j R^j)| \leqslant C\epsilon_i k_i ||R|| \, ||S|| \quad (1 \leqslant j \leqslant n). \tag{18}$$

Then, combining (16)–(18) we see that

$$|b_T(R,S)| \leqslant 2C\left(\sum_i \epsilon_i\right) ||R|| \, ||S|| \quad (R, S \in \mathfrak{A}).$$

Since $\mathfrak{A}$ is dense in $\mathscr{F}(X)$ in the projective norm, it follows that $T \in \Delta_X$ as desired.

It remains to show that $\lim_n |\mathrm{tr}((T-\lambda)P_n)| = \infty$ for all $\lambda \in \mathbb{C}$. For this first note that the following identity holds:

$$\mathrm{tr}((T-\lambda)P_n) = n - \lambda\left(\sum_{i=1}^{n} k_i\right) \quad (n \in \mathbb{N}, \ \lambda \in \mathbb{C}).$$

Since $\sum_{i=1}^{\infty} \epsilon_i < \infty$ we have $\lim_i \epsilon_i = 0$, and in turn $\lim_i \epsilon_i^{-1} = \infty$. Thus $\lim_n \frac{1}{n}\sum_{i=1}^{n} \epsilon_i^{-1} = \infty$. Then taking into account the fact that $\sum_{i=1}^{n} k_i \geqslant \sum_{i=1}^{n} \epsilon_i^{-1}$ $(n \in \mathbb{N})$, it follows that $\lim_n |n - \lambda(\sum_{i=1}^{n} k_i)| = \infty$ for all $\lambda \in \mathbb{C}$ as we wanted. □

To produce families of non-isomorphic Banach spaces with the same property we combine the previous construction with one given by Figiel [Fi]. Let us recall the following result from [Fi].

Proposition 5.4. *Let (q_n) be a strictly decreasing sequence of real numbers greater than 2, and let (m_n) and (k_n) be sequences of positive integers such that*

1. *for every m_n-dimensional linear subspace Z of $(\sum \oplus_{j=1}^{\infty} l_{q_{n+j}})_2$ the distance $d(Z, l_{q_n}^{m_n}) > n$ $(n \in \mathbb{N})$; and*
2. *$k_n > n\sum_{i<n} k_i$, and each subspace $Y \subseteq l_{q_n}^{nk_n}$ with $\dim Y > \frac{1}{2}k_n$ contains a subspace Z such that $\dim Z = m_n$ and $d(Z, l_{q_n}^{m_n}) < 2$ $(n \in \mathbb{N})$.*

Then the Banach space $X = (\sum \oplus_{n=1}^{\infty} l_{q_n}^{k_n})_2$ has the following property: for any $n \in \mathbb{N}$, there exists no isomorphic embedding of X^{n+1} into X^n (here X^n denotes the Banach space $\sum \oplus_{i=1}^{n} X_i$, where $X_i = X$ $(1 \leqslant i \leqslant n)$).

The existence of sequences (m_n) and (k_n) as in Proposition 5.4 is also proved in [Fi]. We show next how sequences (p_n), (k_n) and (m_n) can be defined such that (p_n) and (k_n) satisfy (15), and (q_n), (k_n) and (m_n), where $q_n := p_n/(p_n - 1)$ $(n \in \mathbb{N})$, satisfy the hypotheses of Proposition 5.4.

Let (ϵ_n) be a sequence of positive numbers such that $\sum_n \epsilon_n < \infty$. We define (p_n), (k_n), and (m_n) in an inductive way. First we choose $p_1 \in (1, 2)$ arbitrarily. Then, assuming that $p_1, p_2, \ldots, p_n$, and $k_1, k_2, \ldots, k_{n-1}$ have already been defined, we define k_n, m_n and p_{n+1} as follows:

1. Choose $\bar{p}_n \in (p_n, 2)$;
2. Let $q_n = p_n/(p_n - 1)$ and $\bar{q}_n = \bar{p}_n/(\bar{p}_n - 1)$, that is, q_n and $\bar{q}_n$ are the conjugates of p_n and $\bar{p}_n$, respectively. Choose $m_n > (nL(\bar{q}_n))^{q_n\bar{q}_n/(q_n - \bar{q}_n)}$ (here $L(\bar{q}_n)$ as in the proof of [Fi, Proposition A]);
3. Choose $k_n \in \mathbb{N}$, $k_n > k_{n-1}$ such that
   (a) $k_n^{\frac{1}{p_n} - \frac{1}{2}} \geqslant \epsilon_n^{-1}$;
   (b) $k_n > n \sum_{i<n} k_i$; and
   (c) each subspace $Y \subseteq l_{q_n}^{nk_n}$ with $\dim Y > \frac{1}{2}k_n$ contains a subspace Z such that $\dim Z = m_n$ and $d(Z, l_{q_n}^{k_n}) < 2$. (by [Fi, Proposition 3], k_n exists);
4. Choose $p_{n+1} \in (\bar{p}_n, 2)$ such that $k_n^{\frac{1}{p_{n+1}} - \frac{1}{2}} \leqslant 2$.

Clearly, sequences (p_n) and (k_n) defined above are both strictly increasing, and satisfy inequalities (15). It is also clear that (q_n) is a strictly decreasing sequence of real numbers in the interval $(2, \infty)$, and that (m_n) and (q_n) satisfy condition (2) of Proposition 5.4 (the latter is condition 3(c) in the above definition). To see that condition (1) is satisfied, first note that, since $\bar{q}_n > q_{n+1}$, the modulus of convexity of $(\sum \oplus_{j=1}^{\infty} l_{q_{n+j}})_2$ admits the estimate $K(\bar{q}_n)\varepsilon^{\bar{q}_n}$ $(0 \leqslant \varepsilon \leqslant 2)$, where $K(\bar{q}_n)$ is a constant independent of ε (see [Fi, Proposition 1] and corollary after it). Then since $q_n > \bar{q}_n$, it follows from the proof of [Fi, Proposition A] and our definition of m_n that condition (1) of Proposition 5.4 is also satisfied.

Define $X := (\sum \oplus_n l_{p_n}^{k_n})_2$. We have seen already that for (p_n) and (k_n) as above, $\mathcal{A}(X)$ is not weakly amenable. Moreover, by Proposition 3.10, the same is true for $\mathcal{A}(X')$. As $X' = (\sum \oplus_n l_{q_n}^{k_n})_2$ we have by Proposition 5.4 that no two different positive powers of X' are isomorphic. Combining this result with the remark after Corollary 3.9 we obtain that the sequence (X_n) defined by $X_n := (X')^n$ $(n \in \mathbb{N})$, is a sequence of pairwise non-isomorphic Banach spaces with unconditional basis, and such that $\mathcal{A}(X_n)$ is not weakly amenable $(n \in \mathbb{N})$.

If we consider subspaces of the Banach space X defined in the last paragraph, instead of direct sums, it is possible to generate an uncountable family of Banach spaces with the same properties as those of the sequence of the previous paragraph. This is the content of our last proposition.

Let us denote by $\mathscr{S}$ the family of all strictly increasing sequences in $\mathbb{N}$, and given $(n_i)\in\mathscr{S}$, let us write $\{(n_i)\}$ for the set of its elements. We use the following result from [CKL].

Lemma 5.5. *Let (q_n), (m_n) and (k_n) be sequences as in Proposition 5.4, and let $(n_i),(l_i)\in\mathscr{S}$. Then $(\sum\oplus_i l_{q_{n_i}}^{k_{n_i}})_2\simeq(\sum\oplus_i l_{q_{l_i}}^{k_{l_i}})_2$ if and only if the symmetric difference $\{(n_i)\}\Delta\{(l_i)\}$ is finite.*

Proposition 5.6. *There exists an uncountable family $(X_\alpha)_{\alpha\in I}$ of pairwise non-isomorphic Banach spaces, each with an unconditional basis, and such that $\mathscr{A}(X_\alpha)$ is not weakly amenable $(\alpha\in I)$.*

Proof. Let $X=(\sum\oplus_i l_{p_n}^{k_n})_2$ be as in the discussion above. Let I be an uncountable subset of $\mathscr{S}$ such that for any two sequences $(n_i),(l_i)\in I$, the symmetric difference $\{(n_i)\}\Delta\{(l_i)\}$ is infinite. Define $X_{(n_i)}:=(\sum\oplus_i l_{q_{n_i}}^{k_{n_i}})_2$ $((n_i)\in I)$. Plainly, each $X_{(n_i)}$ $((n_i)\in I)$ has an unconditional basis. Next note that $X_{(n_i)}$ $((n_i)\in I)$ is not weakly amenable, for $X_{(n_i)}$ is the dual of $(\sum\oplus_i l_{p_{n_i}}^{k_{n_i}})_2$, and it is easily seen that the (sub)sequences (p_{n_i}) and (k_{n_i}) (of (p_n) and (k_n) respectively) satisfy the inequalities (15), the same as (p_n) and (k_n). Finally, by Lemma 5.5, and our choice of I we have that no two different members of the family $(X_{(n_i)})_{(n_i)\in I}$ are isomorphic. $\square$

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